Indian Statistical Institute, Bangalore Centre
B.Math. (I Year) : 2010-2011
Semester II : Semestral Examination
Probability Theory II

25.4.2011 Time: 3 hours. Maximum Marks : 100

Note: The paper carries 105 marks. Any score above 100 will be taken as 100. State clearly the results you are using in your answers.

1. [9+6+10=25 marks] Let $g(\cdot)$ denote the probability density function of the one dimensional standard normal distribution. Let $u(\cdot)$ be a non-trivial continuous function on $\mathbb{R}$ such that (a) $u(-z) = -u(z)$ for all $z \in \mathbb{R}$; (b) $|u(z)| \leq \frac{1}{2}\frac{1}{\sqrt{2\pi e}}$ for all $z \in \mathbb{R}$; (c) $u(z) = 0$ for all $z \in (-1,1)^c$. Define

$$f(x,y) = g(x)g(y) + u(x)u(y), \quad (x,y) \in \mathbb{R}^2.$$

(i) Show that $f(\cdot,\cdot)$ is a 2-dimensional probability density function.

(ii) Let (X,Y) be a 2-dimensional random variable with $f(\cdot,\cdot)$ as its probability density function. Find the marginal density functions. Are X, Y independent?

(iii) Show that (X,Y) does not have a bivariate normal distribution. (*Hint:* If (X,Y) is bivariate normal, first prove that

$$f(x,y) = K \exp\left(-(ax^2 + bxy + cy^2)\right),$$

where K, a, c are positive constants.)

2. [15 marks] Suppose that the weight W of a person selected at random from a population has $N(\mu,\sigma^2)$ distribution. Suppose that in recording W an error T is made; assume that T has a uniform distribution over $(-1,1)$, and that W, T are independent. Show that the distribution function of the actually recorded weight X is given by

$$F_X(x) = \frac{1}{2}\int_{-1}^{1} \Phi(\frac{x-t-\mu}{\sigma})dt, \quad x \in \mathbb{R},$$

where $\Phi(\cdot)$ denotes the distribution function of the standard normal distribution.

3. [10+3+7=20 marks] Let $X_1, X_2, \ldots, X_n$ be independent random variables, each having an exponential distribution with parameter $\lambda > 0$. Define $Y_k = \sum_{j=1}^{k} X_j$, for $k = 1, 2, \ldots, n$.

(i) Find the joint probability density function of $Y_1, Y_2, \ldots, Y_n$.

(ii) What is the marginal distribution of Y_k for $1 \leq k \leq n$?

(iii) Find the probability density function of (W, Z), where

$$W = \min\{X_1, X_2, \ldots, X_n\}, \;\; Z = \max\{X_1, X_2, \ldots, X_n\}.$$

4. [9+6=15 marks] (i) Let the random variable X have probability density function $f(x) = \frac{1}{2}\exp(-|x|), \;\; x \in \mathbb{R}$. Find its characteristic function $\varphi(\cdot)$, and show that $\varphi(\cdot)$ is integrable.

(ii) Using (i) find the characteristic function of the Cauchy distribution.

5. [15 marks] 300 numbers are rounded off to the nearest integer and then added. Assume that the individual round-off errors are independent and uniformly distributed over $(-0.5, 0.5)$. Find the probability that the computed sum will differ from the sum of the original 300 numbers by more than 10.

6. [15 marks] Let $X_1, X_2, \ldots$ be independent standard normal random variables; let $\Phi(\cdot)$ denote their common distribution function. Show that for any real number x

$$\lim_{n\to\infty} P\{\Phi(X_1) + \Phi(X_2) + \cdots + \Phi(X_n) \leq \frac{n}{2} + \sqrt{\frac{n}{12}}x\} = \Phi(x).$$